C24 Dynamical Systems

Lecture 1: Introduction

Mark Cannon

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mark.cannon@eng.ox.ac.uk

Overview

- In this course we will study the topological properties of solutions of ordinary differential equations – this can be done without explicitly solving them
- Essentially about the geometry of the paths describing the evolution of solutions over time, and how these paths can be thought of as lying on 'surfaces'
- Intimately associated with the idea of state or phase space and how solutions are related to the 'state'

Course Summary

1. Introduction to dynamical systems
2. Phase space and equilibria
3. Stable, unstable and centre subspaces
4. Lyapunov functions, gradient & Hamiltonian systems, vector fields possessing an integral
5. Invariance, La Salle's theorem and domains of attraction
6. Limit sets, attractors, orbits, limit cycles, Poincaré maps
7. Saddle-node, transcritical, pitchfork and Hopf bifurcations
8. Logistic map, fractals and chaos, Lorenz equations

Course structure

- 8 Lectures in LR2
  - 14:00 Wednesday
  - 12:00 Friday

- 2 examples classes at the beginning of HT26

- Lecture notes, slides & lecture recordings available on Canvas

Example dynamical system: logistic equation

- The logistic equation models population growth of a single species in an environment:

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

$x:$	population at time t
$b > 0:$	birth rate
$K > 0:$	carrying capacity of the environment

- Solution (see lecture notes):

$$\left.\begin{aligned} t &= \tau/b \\ x &= K\theta \end{aligned}\right\} \implies \frac{d\theta}{d\tau} = \theta(1-\theta) \implies \int_{\theta_0}^{\theta} \frac{d\theta}{\theta(1-\theta)} = \int_0^{\tau} d\tau \text{ or } \ln\left[\frac{\theta(1-\theta_0)}{(1-\theta)\theta_0}\right] = \tau$$

$$\left.\begin{aligned} \tau &= bt \\ \theta &= x/K \end{aligned}\right\} \implies x = \frac{Kx_0e^{bt}}{K + x_0(e^{bt}-1)} \text{ so } x_0 = \frac{cK}{c+K} \implies \boxed{x(t) = \frac{Kce^{bt}}{K + ce^{bt}}}$$

The logistic equation and its solution

$$\frac{dx}{dt} = bx(1 - \frac{x}{K}) \quad \text{has the solution} \quad x(t) = \frac{Kce^{bt}}{K + ce^{bt}}$$

- We have an analytic solution, but what does it say?
  Is it informative to have this exact answer?

- What happens if $x(0) = 0$? and what does this signify?

- What happens as $t \to \pm\infty$?

- Can we analyse these properties without solving the equation?

To address these questions we introduce geometric ideas into the problem

Phase (or state) space

- An nth order ordinary differential equation (ODE) in a single variable $x(t)$ can be written as n coupled 1st order differential equations in $x_1(t), x_2(t), \ldots, x_n(t)$

For example $$\frac{d^2x}{dt^2} - b\frac{dx}{dt} + cx = g$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx_1}{dt} \end{aligned}\right\} \implies \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

- Each variable $x_1, x_2, \ldots, x_n$ defines a coordinate in phase (or state) space

- Solutions of the ODE create curves (or trajectories) in phase (or state) space, which are determined by the intial conditions

Phase space: terminology

- We use concepts from geometry to name phase spaces:
  - if $n = 1$ (1st order ODE) we have a phase line
  - if $n = 2$ (2nd order ODE) we have a phase plane
  - if $n > 2$ we have a general phase space
- We will find that collections of similar trajectories that solve a problem form surfaces, which are sometimes called solution manifolds (smooth surfaces)
- Recall that this is not totally new: phase space/state space was introduced in P1 Mathematical modelling and A2 Introduction to Control Systems

Back to the logistic equation example

- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

- This equation is 1st order $(n = 1)$ so the solution trajectories lie on a phase line

- There are special points where the solution remains stationary (i.e. $x(t)$ does not change with time):

$$\frac{dx}{dt} = 0 \implies bx\left(1 - \frac{x}{K}\right) = 0 \implies x = 0 \text{ or } K$$

- These special points are called equilibrium points (or fixed points) of the system

Logistic equation on the phase line

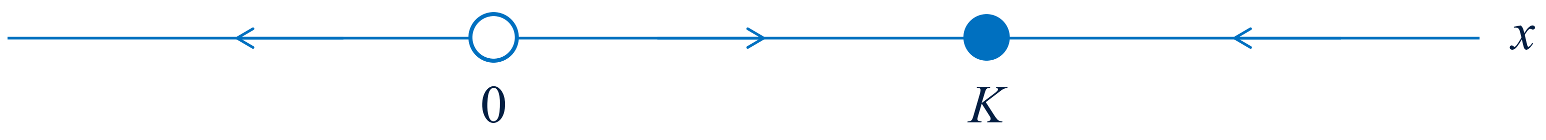
- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

- Instead of solving this (to find how x varies with t), consider how the rate of change of x depends on x

$$x < 0 \implies \frac{dx}{dt} < 0$$

$$0 < x < K \implies \frac{dx}{dt} > 0$$

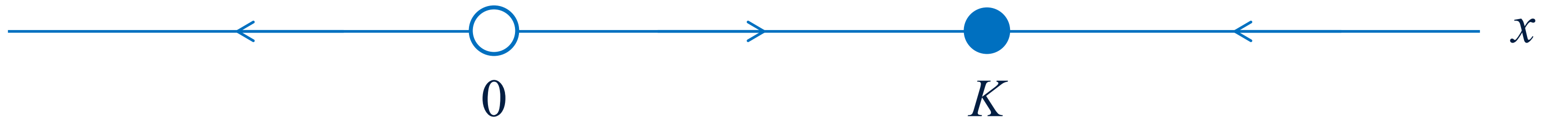
$$x > K \implies \frac{dx}{dt} < 0$$



phase portrait of the logistic equation

Stable and unstable equilibria

- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$



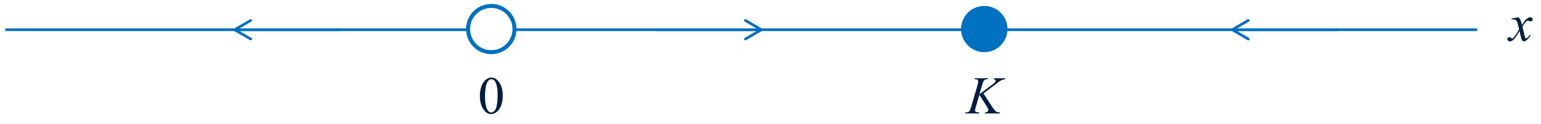
- All trajectories near $x = 0$ move away from that equilibrium point: we call such an equilibrium unstable

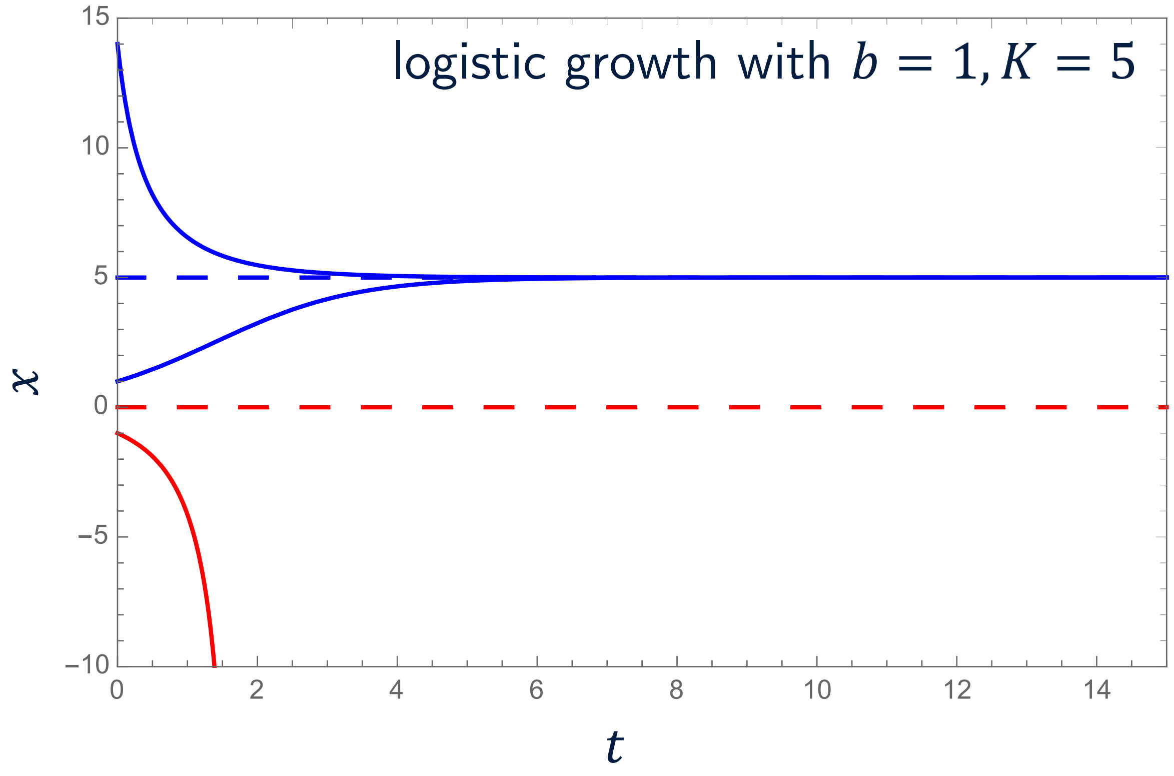
- All trajectories near $x = K$ flow toward that equilibrium point: such an equilibrium is called stable

- Note also that it is not possible to move from values $x < K$ to values $x > K$ without crossing $x = K$; this makes trajectories stop at $x = K$ So there is no overshoot at $x = K$

Logistic equation: visualizing solutions

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$





Example: damped single pendulum

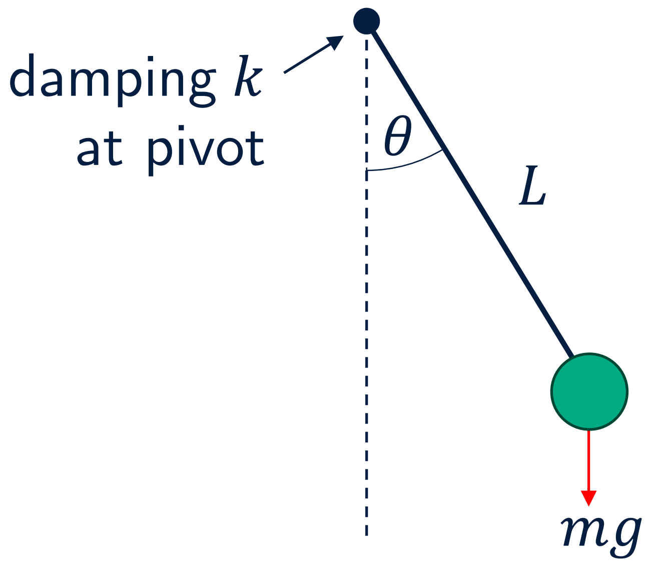
- $mL\frac{d^2\theta}{dt^2} = -mg\sin\theta - kL\frac{d\theta}{dt}$

- Define $\tau = t\sqrt{\frac{g}{L}}$, $b = \frac{k}{m}k\sqrt{\frac{L}{g}}$, then

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta - b\frac{d\theta}{d\tau}$$

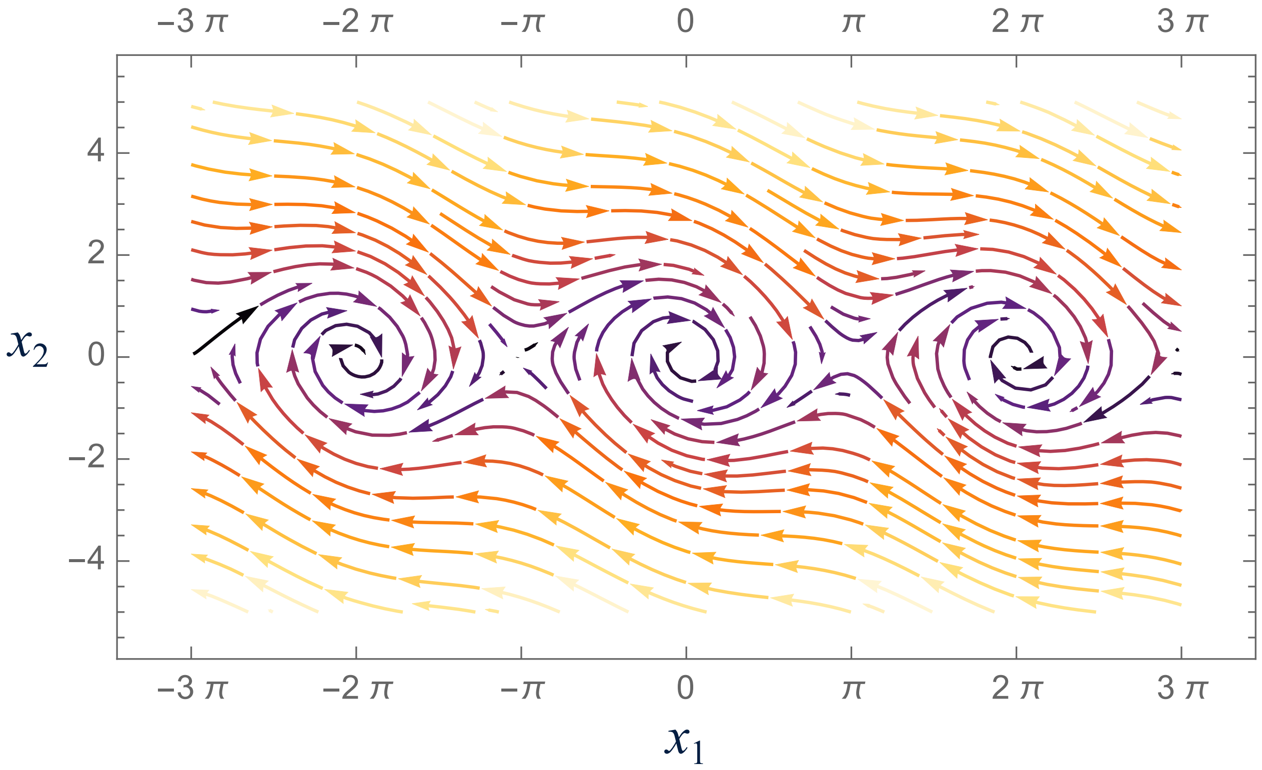
- 2nd order, so two states; let $x_1 = \theta$ and $x_2 = \frac{d\theta}{d\tau}$:

$$\frac{dx_1}{d\tau} = x_2$$
$$\frac{dx_2}{d\tau} = -\sin x_1 - bx_2$$



Phase plane for damped pendulum

- Equilibrium points at $x_2 = \sin x_1 = 0$
- Phase plane of the damped pendulum, $b = 1/\sqrt{10}$



$$\frac{dx_1}{d\tau} = x_2$$
$$\frac{dx_2}{d\tau} = -\sin x_1 - bx_2$$

Example: glycolitic oscillations

- Glycolysis is a process in which glucose is turned into energy compounds like ATP inside cells

$$\frac{dx}{dt} = -x + ay + x^2 y$$
$$\frac{dy}{dt} = b - ay - x^2 y$$

$x(t)$ and $y(t)$ represent concentrations of reaction intermediates

- Equilibrium conditions: $\begin{cases} x = b \\ y = \dfrac{b}{a+b^2} \end{cases}$ so let $\begin{cases} x_1 = \dfrac{x}{b} \\ x_2 = \dfrac{(a+b^2)}{b} y \end{cases}$

then
$$\frac{dx_1}{dt} = -x_1 + \left(\frac{a}{a+b^2}\right) x_1^2 x_2$$
$$\frac{dx_2}{dt} = (a+b^2)\left[1 - \left(\frac{a}{a+b^2}\right) x_2 - \left(\frac{b^2}{a+b^2}\right) x_1^2 x_2\right]$$

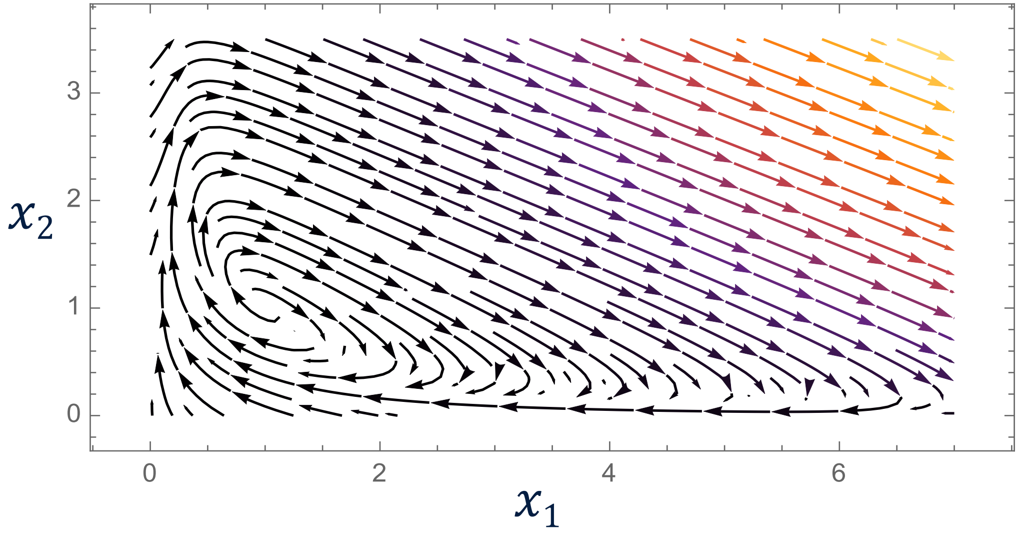
Phase plane for glycolytic oscillations

$$\frac{dx_1}{dt} = -x_1 + \frac{a}{a+b^2}x_2 + \frac{b^2}{a+b^2}x_1^{\,2}x_2$$

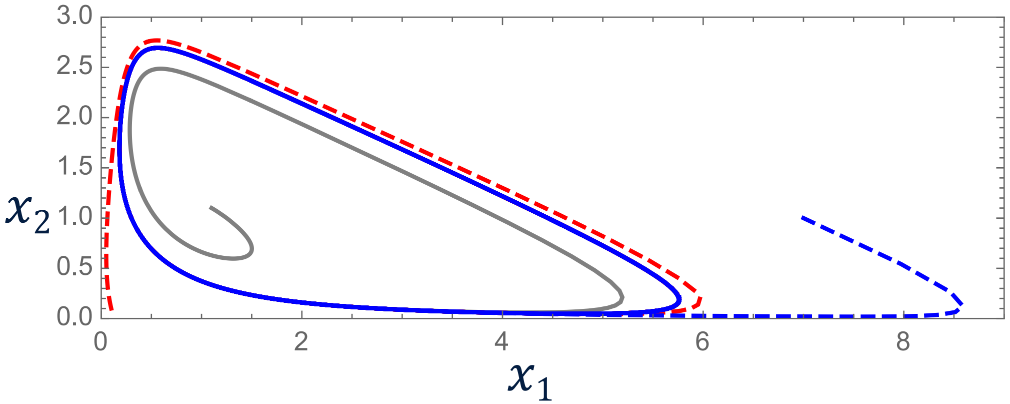
$$\frac{dx_2}{dt} = \left(a+b^2\right)\left[1 - \frac{a}{a+b^2}x_2 - \frac{b^2}{a+b^2}x_1^{\,2}x_2\right]$$

- Study oscillations with $a = 0.03$, $b = 0.6$

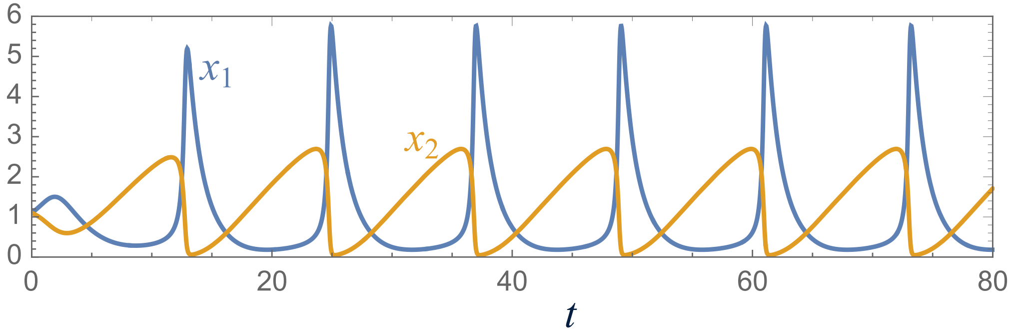
- Phase plane



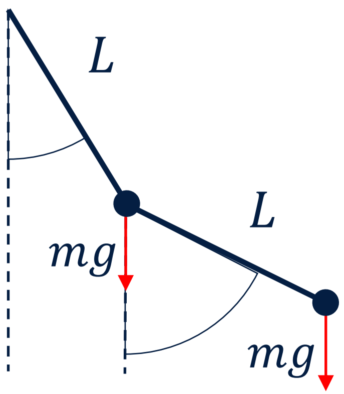
- Trajectories for three initial conditions: $(x_1(0), x_2(0)) = (0.1, 0.1), (7, 1), (1.1, 1.1)$

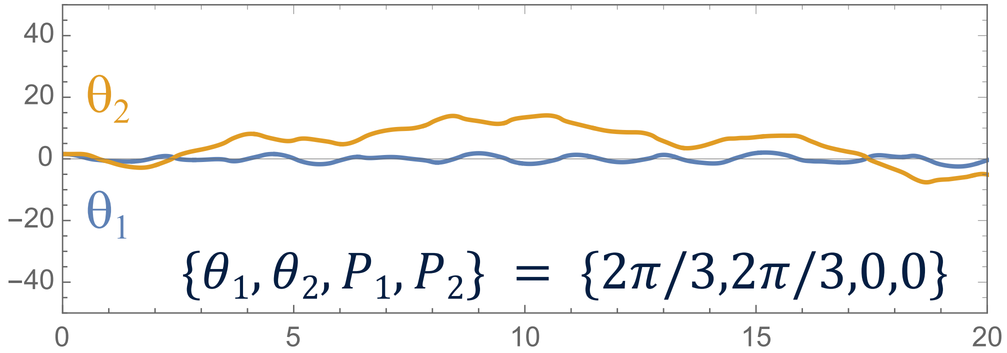


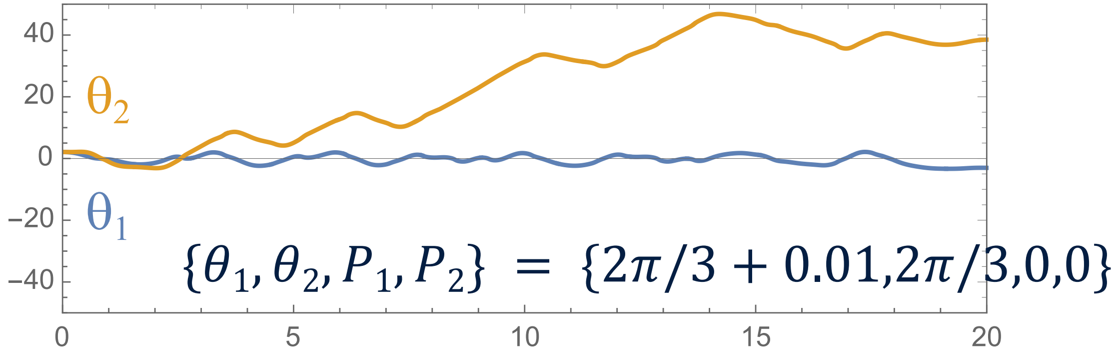
- Time response with $(x_1(0), x_2(0)) = (1.1, 1.1)$:



Example: double pendulum







$$\frac{d\theta_1}{dt} = \frac{P_1 - P_2\cos(\theta_1-\theta_2)}{1+\sin^2(\theta_1-\theta_2)}$$

$$\frac{d\theta_2}{dt} = \frac{2P_2 - P_1\cos(\theta_1-\theta_2)}{1+\sin^2(\theta_1-\theta_2)}$$

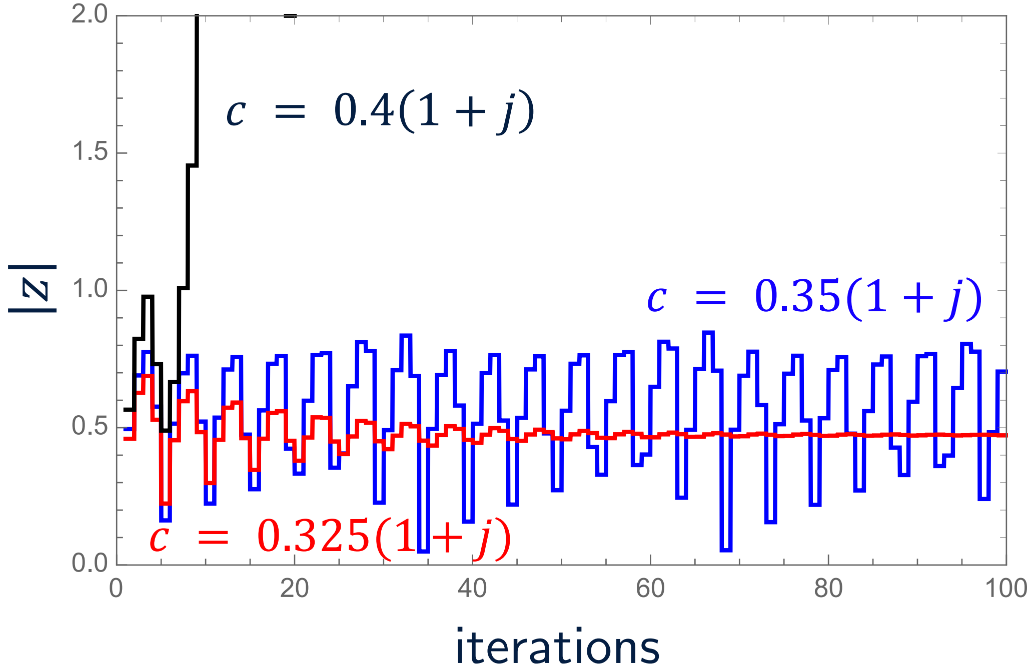
$$\frac{dP_1}{dt} = -2\frac{g}{L}\sin\theta_1 - \frac{\left[2P_1P_2 - \left(P_1^2+2P_2^2\right)\cos(\theta_1-\theta_2)\right]\sin(\theta_1-\theta_2)}{\left[1+\sin^2(\theta_1-\theta_2)\right]^2}$$

$$\frac{dP_2}{dt} = -\frac{g}{L}\sin\theta_2 + \frac{\left[2P_1P_2 - \left(P_1^2+2P_2^2\right)\cos(\theta_1-\theta_2)\right]\sin(\theta_1-\theta_2)}{\left[1+\sin^2(\theta_1-\theta_2)\right]^2}$$

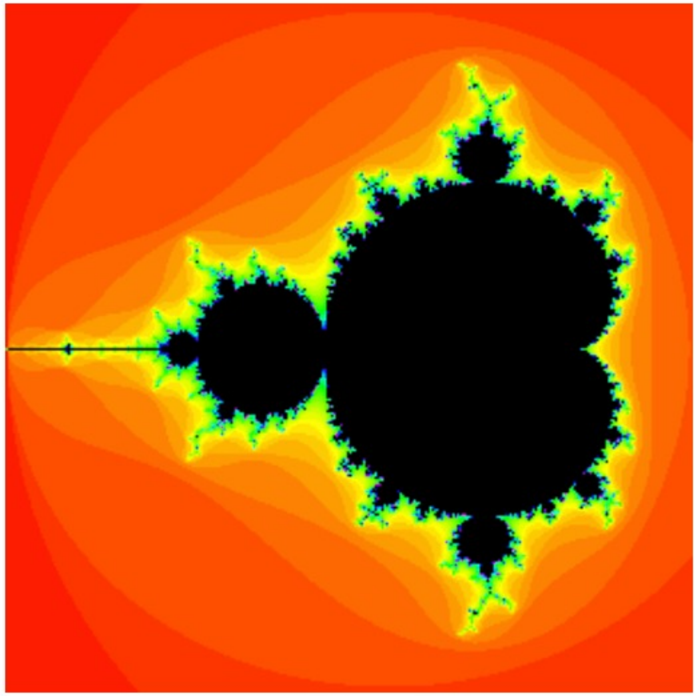
The two solutions start very near the same point in phase space, but the transients differ dramatically – why?

Example: the Mandelbrot set

- An iterative equation: $z_{k+1} = z_k^2 + c$ with c a complex number
- Question: if $z_0 = 0$, for which values of c does $|z_k|$ remain bounded?
- Effect of varying c values



- $|z_k|$ remains bounded for c in the black set



Strategy for analyzing dynamical systems

- Start by finding equilibrium points
- The nature of each equilibrium point can often be established by investigating the local (Jacobian) linearization of the dynamics
- We then study the geometry and topology (connectedness) of regions around equilibria in phase space
- We reason about the flow of solution trajectories through these regions
- To implement this strategy we will use topics from linear algebra – eigenvalues and eigenvectors of matrices

Eigenvalues and eigenvectors review

- Suppose $\mathbf{A}$ is square matrix that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$
- An eigenvalue λ and corresponding eigenvector $\mathbf{v}$ of $\mathbf{A}$ satisfy $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
- Hence the eigenvalues are the n roots of the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

- Recall that complex eigenvalues appear in conjugate pairs
- Further recall that if all the eigenvalues are distinct, then there is a complete set of n linearly independent eigenvectors

Eigenvectors as a basis

- If $\mathbf{A}$ has a complete set of n real eigenvectors, then they will span $\mathbb{R}^n$
- Hence the set of eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$
- Put another way, any vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as a linear combination of the eigenvectors in this case:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

- If the eigenvalues of a matrix are not distinct (i.e. the characteristic equation has a repeated root) then there may not be a complete set of eigenvectors, but there will be a set of 'generalized eigenvectors' (Perko Ch.1)

Matrix diagonalization

- If $\mathbf{A}$ has n distinct eigenvalues, then there is a complete set of n eigenvectors that spans $\mathbb{R}^n$
- In this case $\mathbf{A}$ is diagonalizable, that is, there exists an invertible matrix $\mathbf{V}$ and diagonal matrix $\mathbf{\Lambda}$ such that
$$\mathbf{AV} = \mathbf{V\Lambda} \quad \implies \mathbf{A} = \mathbf{V\Lambda V}^{-1}$$
- The kth diagonal entry of $\mathbf{\Lambda}$ is the kth eigenvalue of $\mathbf{A}$, the kth column of $\mathbf{V}$ is the corresponding eigenvector

Complex eigenvalues

- If $\mathbf{A}$ has complex eigenvalues, then its eigenvectors will be complex, so $\mathbf{A}$ cannot be diagonalized using a matrix $\mathbf{V}$ of real numbers

- There is a simple way to rearrange the diagonalization in the case of 2×2 matrices with complex eigenvalues:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \Longrightarrow \quad \begin{array}{l} \lambda \in \{a + jb,\, a - jb\} \\ \mathbf{v} \in \{\mathbf{u} + j\mathbf{w},\, \mathbf{u} - j\mathbf{w}\} \end{array}$$

- Let $\mathbf{V}' = [\mathbf{w} \quad \mathbf{u}]$, then we can write the standard form

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}' \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{V}'^{-1}$$

Example

- Consider $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ with eigenvalues: $\lambda_1 = 2 + j$, $\lambda_2 = 2 - j$

- eigenvectors are solutions of: $\begin{bmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\implies \quad \mathbf{v}_1 = \begin{bmatrix} 1+j \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1-j \\ 1 \end{bmatrix}$$

- so $\lambda = a \pm jb$, $\mathbf{v} = \mathbf{u} \pm j\mathbf{w}$ with $(a, b) = (2, 1)$, $(\mathbf{u}, \mathbf{w}) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1+j & 1-j \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+j & 0 \\ 0 & 2-j \end{bmatrix} \begin{bmatrix} 1+j & 1-j \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \end{aligned}$$

Linear autonomous systems

- An autonomous system of first-order differential equations depends on the dependent variables, but does not explicitly include the independent variable:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \qquad \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

autonomous | not autonomous

- The linear autonomous system of first-order differential equations can be written in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

- Define $e^{\mathbf{A}} \triangleq \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$ then if $\mathbf{A}$ is diagonalizable

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$$

Computation of matrix exponentials

- To compute the matrix exponential, we use diagonalization:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right)^k}{k!} = \mathbf{V}\left(\sum_{k=0}^{\infty} \frac{\boldsymbol{\Lambda}^k}{k!}\right)\mathbf{V}^{-1} = \mathbf{V}\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}\mathbf{V}^{-1}$$

- This is straightforward if the eigenvalues are real and distinct

- For complex eigenvalues, we can use the standard form of a 2×2:

since $\quad \lambda = a \pm jb \quad \implies \quad e^{\lambda} = e^{a}(\cos b \pm j \sin b)$

we get $\quad e^{\mathbf{A}} = \mathbf{V}' \begin{bmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{bmatrix} \mathbf{V}'^{-1}$

- See Perko *Differential equations and dynamical systems*, sec 1.5

Definitions for dynamical systems 1

- Generally, ordinary differential equations of order n can be represented instead as a set of n coupled 1st-order ODEs

- Each of the n dependent variables in this ODE system is called a **state**, $x_i(t) \in \mathbb{R}$, and $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector

- Generally we can write the ODE system as a list of functions $\dot{x}_i = f_i(\mathbf{x})$ where f_i is a mapping from the vector space of states into the real numbers, i.e. $f_i : \mathbb{R}^n \to \mathbb{R}$

- If f_i is defined on a subspace $D \subset \mathbb{R}^n$, called the **domain** of the function, then we write $f_i : D \to \mathbb{R}$
(e.g. the square root function is limited to $D = \{x : x \geq 0\}$)

- We let $\mathbf{f}$ represent the vector whose ith entry is f_i, so $\mathbf{f} : D \to \mathbb{R}^n$

Definitions for dynamical systems 2

- The general set of nonlinear **autonomous** systems can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

- Systems can have **parameters** so $\mathbf{f}$ may depend on a vector $\mathbf{p} \in \mathbb{R}^p$:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; \mathbf{p}) \implies \frac{d\mathbf{x}}{dt} \text{ is a function } \mathbf{f} \text{ of } \mathbf{x} \text{ parameterised by } \mathbf{p}$$

- We can also consider **difference equations** (also called maps), which are not ODEs but follow recurrence relations:

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k; \mathbf{p})$$

Some notation for the solutions of dynamical systems

- Given an autonomous ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x};\mathbf{p})$$

- A solution of the ODE is a map from the time interval $t \in (\alpha, \beta)$ into the space $\mathbb{R}^n$, which passes through initial condition $\mathbf{x}_0 \in \mathbb{R}^n$

$$\mathbf{x} : (\alpha, \beta) \to \mathbb{R}^n \text{ such that } \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t);\mathbf{p}) \text{ and } \mathbf{x}(0) = \mathbf{x}_0$$

- Here we will not be concerned with solving such equations – instead we will look at the geometry of these solutions

Existence and uniqueness of solutions

- Does a solution **exist?** Is it **unique?**

- The study of existence and uniqueness is quite technical…

- The lecture notes describe some conditions ensuring existence based on Lipschitz continuity, see

  - lecture notes sec. 1.3.1
  - Perko sec. 2.2 & 2.3

  This is for interest only (non-examinable)

Questions?